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ON THE EXPONENTIAL CONVERGENCE OF THE h-p VERSION FOR BOUNDARY ELEMENT GALERKIN METHODS ON POLYGONS

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ABSTRACT

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On the Exponential Convergence of the h-p Version for Boundary Element Galerkin Methods on Polygons

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1. Introduction

Most research for the boundary element method (BEM) has been carried out in the framework of the h-version [11], [13], [28] where accuracy is achieved by decreasing the mesh size h, while keeping the degree p of piecewise polynomials fixed. For this method several detailed results, including asymptotic rates of convergence for both first-kind and second-kind integral equations are well-known (see [21], [28]). The basic idea of the above convergence proofs is the observation that for strongly elliptic operators one obtains quasioptimal convergence in the energy norm for any Galerkin scheme with conforming boundary elements (see [27]). This result has been used in [25] and [26] to analyze the p- and h-p versions with quasiuniform mesh for BEM for some first kind integral equation on a polygon Γ. Those versions for BEM have been recently introduced (see [1], [2], [18], [22], [23]). In the p-version a fixed mesh with constant h is used, and accuracy is achieved by increasing the degree p of the polynomials used as boundary elements. The h-p version combines the two approaches. If one uses a quasiuniform mesh on the polygon Γ and if the singularity of the solution of the integral equation is located at vertices, the rate of convergence for the p-version of the boundary element method (BEM) is twice that of the h-version (see [25], [26]) for some first kind integral equations. These results have been known for the finite element method where these two extension processes (p- and h-p version) have been thoroughly investigated in a series of papers [3], [9], [10], [8], [14]. Furthermore, for the finite element method on a geometric mesh it has been shown in [17] that under proper assumptions satisfied usually in practice, namely that the given data are piecewise analytic, the h-p version has an exponential rate of convergence with respect to the number of degrees of freedom while the h- and p-versions have only a polynomial rate. In this paper we show the corresponding result for the boundary element method, i.e., if one uses a geometric mesh Γ_σ^n on the boundary Γ which is graded towards the vertices, then the convergence rate of the Galerkin solution for the underlying boundary integral equation is exponential. We consider both the Dirichlet and the Neumann problem for the Laplacian in a plane polygonal domain and solve them via first-kind integral equations on Γ with a weakly singular and hypersingular integral operator V and D, respectively. It is known [13] that the operator V of the single layer potential is a continuous, bijective mapping from the Sobolev space $H^{-1/2}(\Gamma)$ onto $H^{1/2}(\Gamma)$ if for the conformal radius of the polygon Γ there holds $\mathrm{Cap}(\Gamma) \neq 1$. From [12] we have that the operator D of the normal derivative of the double layer potential maps $H^{1/2}(\Gamma)$ continuously and bijectively onto $H^{-1/2}(\Gamma)$. As one of our main results we show that if the given data are piecewise analytic then the solutions of the integral equations are also piecewise analytic, i.e., for given data in some countable normed space $g \in B_{\beta}^{0,1}(\Gamma)$ we have $u \in B_{\beta}^{1,2}(\Gamma)$ satisfies

$$Du = (1 - K')g \text{ on } \Gamma, \tag{1.1}$$

whereas for $g \in B_{\beta}^{1,2}(\Gamma) \cap C^0(\Gamma)$ we have $\frac{\partial u}{\partial n} \in B_{\beta}^{0,1}(\Gamma)$ satisfies

$$V \frac{\partial u}{\partial n} = (1 + K)g \text{ on } \Gamma, \tag{1.2}$$

where K, K' are the operators of the double layer potential and its adjoint.

In Section 2 we introduce the weighted Sobolev spaces $H_{\beta}^{k,\ell}(\Omega)$ $(k \geq \ell \geq 0)$ and the countable normed spaces $B_{\beta}^{\ell}(\Omega)$. We quote some related trace and extension theorems from [6] together with the existence results of the solution $u \in B_{\beta}^{2}(\Omega)$ of the Neumann and Dirichlet boundary value problem, respectively.

In Section 3 we consider the integral equation (1.1) on Γ governing the Neumann problem and prove the above mentioned existence and regularity result for its solution $u \in B_{\beta}^{1,2}(\Gamma)$. Then we present our geometric mesh Γ_{σ}^{n} on the boundary curve Γ for an L-shaped domain and the corresponding conforming boundary element space $\mathring{S}^{p}(\Gamma_{\sigma}^{n})$ consisting of continuous piecewise polynomials of increasing degree where linears are used on the smallest subinterval, quadratics on the next larger subinterval and so forth. Then we show (Theorem 3.4) that the Galerkin solution $u_p \in \mathring{S}^{p}(\Gamma_{\sigma}^{n})$ for (1.1) converges in $H^{1/2}(\Gamma)$ with exponential rate towards the exact solution.

In Section 4 we present the corresponding results for the Dirichlet problem by showing that the solution $\frac{\partial u}{\partial n}$ of the integral equation (1.2) belongs to $B_{\beta}^{0,1}(\Gamma)$ and that its Galerkin solution $\psi_p \in S^{p-1}(\Gamma_{\sigma}^{n})$ converges exponentially in $H^{-1/2}(\Gamma)$.

2. Preliminary

Let $\Omega \in \mathbf{R}^2$ be a bounded domain whose curvilinear boundary $\partial\Omega$ is a piecewise analytic curve $\Gamma = \bigcup_{i=1}^{M} \bar{\Gamma}_i$, where Γ_i is an open arc connecting the vertices A_i and A_{i+1} $(A_1 = A_{M+1})$. We denote the internal angle at A_i by ω_i and assume $0 < \omega_i < 2\pi$, $1 \leq i \leq M$.

Let $H^k(\Omega)$, $k \geq 0$ integer, denote the usual Sobolev space (see [20]), i.e.,

$$H^k(\Omega) = \{u \mid \sum_{0\le|\alpha|\le k} ||D^\alpha u||^2_{L_2(\Omega)} = ||u||^2_{H^k(\Omega)} < \infty\}$$

where $\alpha = (\alpha_1, \alpha_2)$, $\alpha_i \ge 0$ integer, $i = 1,2$, $|\alpha| = \alpha_1 + \alpha_2$, and

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}} = u_{x_1^{\alpha_1} x_2^{\alpha_2}}.$$

Since Ω is a Lipschitz domain any $u \in H^k(\Omega)$ can be extended to $H^k(\mathbf{R}^2)$ (see [24]). Therefore $H^{k-1/2}(\Gamma)$ is defined as the restriction on Γ of functions in $H^k(\Omega)$ for integral $k \ge 1$, i.e.,

$$H^{k-1/2}(\Gamma) = \{u|_\Gamma : u \in H^k(\Omega)\}$$

with

$$||g||_{H^{k-1/2}(\Gamma)} = \inf_{u|_\Gamma = g} ||u||_{H^k(\Omega)}$$

and for $k \le 0$

$$H^{k-1/2}(\Gamma) = (H^{-(k-1/2)}(\Gamma))' \text{ (dual space)}.$$

Let $r_i(x) = \text{dist}(x_i, A_i)$, $\beta = (\beta_1, \beta_2, \ldots \beta_M)$ be an M-tuple of real numbers $0 < \beta_i < 1$. For any integer $k \ge 0$ we shall write $\beta + k = (\beta_1 + k, \beta_2 + k, \ldots, \beta_M + k)$, and $\Phi_{\beta+k}(x) = \prod_{i=1}^{M} r_i^{\beta_i + k}(x)$. We define the weighted Sobolev space for integers k and ℓ, $k \ge \ell \ge 0$, by

$$H^{k,\ell}_\beta(\Omega) = \{u \mid u \in H^{\ell-1}(\Omega) \text{ if } \ell > 0, ||\Phi_{\beta+|\alpha|-\ell} D^\alpha u||_{L_2(\Omega)} < \infty \text{ for } 0 \le \ell \le |\alpha| \le k\}$$

and the countable normed space for $\ell \ge 0$

$$B^\ell_\beta(\Omega) = \{u \in H^{k,\ell}_\beta(\Omega), \forall\, k \ge \ell, ||\Phi_{\beta+k-\ell}\, D^\alpha u||_{L_2(\Omega)} \le Cd^{k-\ell}(k-\ell)!$$

$$\text{for } |\alpha| = k = \ell, \ell+1, \ldots, \text{ with } C \ge 1, d \ge 1 \text{ independent of } k\}$$

The space $H^{k-1/2,\ell-1/2}_\beta(\Gamma)$, (resp. $B^{\ell-1/2}_\beta(\Gamma)$), k, ℓ integer, $k \ge \ell \ge 0$, is the trace space of $H^{k,\ell}_\beta(\Omega)$, (resp. $B^\ell_\beta(\Omega)$), i.e., for any $g \in H^{k-1/2,\ell-1/2}_\beta(\Gamma)$ (resp. $B^{\ell-1/2}_\beta(\Gamma)$) there exists $G \in H^{k,\ell}_\beta(\Omega)$

(resp. $B^{\ell}_{\beta}(\Omega)$) such that $G|_{\Gamma} = g$, and

$$\|g\|_{H_{\beta}^{k-1/2,\ell-1/2}(\Gamma)} = \inf_{G|_{\Gamma}=g} \|G\|_{H_{\beta}^{k,\ell}(\Omega)}.$$

The analogous definition of the weighted Sobolev spaces and countable normed spaces on the interval $I = [a,b]$, $a,b \in \mathbf{R}$, we quote from [6]:

For $k \geq \ell \geq 0$ integer,

$$H^{k,\ell}_{\hat\beta}(I) = \{u \mid u \in H^{\ell-1}(I) \text{ if } \ell > 0,\ \|\hat\Phi_{\hat\beta+m-\ell}\, u^{(m)}(x)\|_{L_2(I)} < \infty \text{ for } 0 \leq \ell \leq m \leq k\}.$$

and for $\ell \geq 0$

$$B^{\ell}_{\hat\beta}(I) := \{u \in H^{k,\ell}_{\hat\beta}(I),\ \forall\ k \geq \ell,\ \|\hat\Phi_{\hat\beta+k-\ell}u^{(k)}(x)\|_{L_2(I)} \leq Cd^{k-\ell}(k-\ell)!$$

$$\text{for } k = \ell, \ell+1, \ldots \text{ with } C \geq 1,\ d \geq 1 \text{ independent of } k\}$$

where $\hat\Phi_{\hat\beta+k-\ell}(x) = \prod_{i=1}^{2} \hat r_i^{\hat\beta_i+k-\ell}(x)$, $\hat r_1(x) = |x-a|$, $\hat r_2(x) = |x-b|$, $\hat\beta = (\hat\beta_1, \hat\beta_2)$, $0 < \hat\beta_1, \hat\beta_2 < 1$.

For any $\Gamma_j \in \Gamma$ the spaces $H^{k,\ell_j}_{\hat\beta_j}(\Gamma_j)$ and $B^{\ell_j}_{\hat\beta_j}(\Gamma_j)$ are defined with the help of a smooth map $I \to \Gamma_j$ via the spaces $H^{k,\ell}_{\hat\beta}(I)$ and $B^{\ell}_{\hat\beta}(I)$. We define $H^{k,\ell}_{\hat\beta}(\Gamma) := \prod_{j=1}^{M} H^{k,\ell_j}_{\hat\beta_j}(\Gamma_j)$ and $B^{\ell}_{\hat\beta}(\Gamma) := \prod_{j=1}^{M} B^{\ell_j}_{\hat\beta_j}(\Gamma_j)$ with $\ell = (\ell_1, \ldots, \ell_M)$ and $\hat\beta = (\hat\beta_1, \hat\beta_2, \ldots, \hat\beta_M)$, $\hat\beta_j = (\hat\beta_{j,1}, \hat\beta_{j,2})$. We shall write $\hat\beta_j \geq \tilde\beta_j$, if $\hat\beta_{j,k} \geq \tilde\beta_{j,k}$, $k = 1,2$, and $\hat\beta \geq \tilde\beta$ if $\hat\beta_j \geq \tilde\beta_j$, $1 \leq j \leq M$. For any real number s we shall write $\hat\beta_j \geq s$ (resp. $\hat\beta \geq s$) if $\hat\beta_{j,k} \geq s$ (resp. $\hat\beta_j \geq s$).

It is difficult to verify whether a function on Γ belongs to the spaces $H_{\beta}^{k-1/2,\ell-1/2}(\Gamma)$ and $B_{\beta}^{\ell-1/2}(\Gamma)$. On the contrary the spaces $H^{k,\ell}_{\hat\beta}(\Gamma)$ and $B^{\ell}_{\hat\beta}(\Gamma)$ characterize the traces of functions of $H^{k,\ell}_{\beta}(\Omega)$ and $B^{\ell}_{\beta}(\Omega)$ on Γ in a precisely verifiable manner (see Theorem 2.1). Theorem 2.2 deals with the extension of functions from Γ to Ω.

Theorem 2.1. (cf. Theorems 4.1 and 4.4 of [6]). *Let* $u \in B^{\ell}_{\beta}(\Omega)$ (resp. $H^{k,\ell}_{\beta}(\Omega)$), $\ell = 1,2$, $k \geq \ell+1$, *then for* $1 \leq i \leq M$, $u|_{\Gamma_i} \in B^{\ell-1}_{\hat\beta_i}(\Gamma_i)$ (resp. $H^{k,\ell-1}_{\hat\beta_i}(\Gamma_i)$) *with*

$$\hat{\beta}_{i,j} \in (\beta_{i+j-1}-\tfrac{1}{2}, \tfrac{1}{2}),\ j = 1,2 \ \text{if}\ \tfrac{1}{2} < \beta_i, \beta_{i+1} < 1 \tag{2.1a}$$

or $u|_{\Gamma_i} \in B^{\ell}_{\hat{\beta}_i}(\Gamma_i)$ (*resp.* $H^{k,\ell}_{\hat{\beta}_i}(\Gamma_i)$) *with*

$$\hat{\beta}_{i,j} \in (\tfrac{1}{2}, \beta_{i+j-1}+\tfrac{1}{2}) \ \text{if}\ 0 < \beta_i, \beta_{i+1} < \tfrac{1}{2}. \tag{2.1b}$$

Theorem 2.2. (cf. Theorems 4.3 and 4.5 of [6]). *Let* g *be defined on* Γ *and* $g_i = g|_{\Gamma_i}$.
(*i*) *If* $g \in C^0(\Gamma)$, *and* $g_i \in B^1_{\hat{\beta}_i}(\Gamma_i)$ (*resp.* $H^{k,1}_{\hat{\beta}_i}(\Gamma_i)$, $k \geq 2$) *for* $0 < \hat{\beta}_i < \frac{1}{2}$ *or* $g_i \in B^2_{\hat{\beta}_i}(\Gamma_i)$ (*resp.* $H^{k,2}_{\hat{\beta}_i}(\Gamma_i)$, $k \geq 2$) *for* $\frac{1}{2} < \hat{\beta}_i < 1$, $1 \leq i \leq M$, *then there exists* $G \in B^2_{\beta^*}(\Omega)$ (*resp.* $H^{k,2}_{\beta^*}(\Omega)$) *such that* $G|_{\Gamma} = g$.
(*ii*) *If* $g_i \in B^0_{\hat{\beta}_i}(\Gamma_i)$ (*resp.* $H^{k,0}_{\hat{\beta}_i}(\Gamma_i)$, $k \geq 1$) *for* $0 < \hat{\beta}_i < \frac{1}{2}$, *or* $g_i \in B^1_{\hat{\beta}_i}(\Gamma_i)$ (*resp.* $H^{k,1}_{\hat{\beta}_i}(\Gamma_i)$) *for* $\frac{1}{2} < \hat{\beta}_i < 1$, $1 \leq i \leq M$, *then there exists* $G \in B^1_{\beta^*}(\Omega)$ (*resp.* $H^{k,1}_{\beta^*}(\Omega)$) *such that* $G|_{\Gamma} = g$.
In (*i*) *and* (*ii*) $\beta^* = (\beta^*_1, \ldots, \beta^*_M)$ *with* β^*_i *satisfying*

$$\beta^*_i = \max(\bar{\beta}_{i-1,2}, \bar{\beta}_{i1}),\ \bar{\beta}_{i,j} = \hat{\beta}_{i,j} - \tfrac{1}{2}\,\text{sign}\,(\hat{\beta}_{i,j}-\tfrac{1}{2}), \tag{2.2}$$

for $1 \leq i \leq M$, $1 \leq j \leq 2$.

In the following sections we assume, for sake of simplicity, that Ω is a straight-line polygon. We will make comments on a curvilinear polygon at the end of the paper. By $B^{\ell,\ell+1}_{\hat{\beta}}(\Gamma)$ we denote for $1 \leq i \leq M$ the space $\prod_{0<\hat{\beta}_i<\frac{1}{2}} B^{\ell}_{\hat{\beta}_i}(\Gamma_i) \times \prod_{\frac{1}{2}<\hat{\beta}_i<1} B^{\ell+1}_{\hat{\beta}_i}(\Gamma_i)$.

In this paper we consider the Dirichlet problem of the Laplace equation

$$\begin{aligned} \Delta u &= 0 \text{ in } \Omega \\ u|_{\Gamma} &= g \end{aligned} \tag{2.3}$$

and the Neumann problem

$$\begin{aligned} \Delta u &= 0 \text{ in } \Omega \\ \frac{\partial u}{\partial n}\Big|_{\Gamma} &= g \end{aligned} \tag{2.4}$$

where $\frac{\partial u}{\partial n}$ means the normal derivative with respect to the unit outer normal n, and g satisfies

$$\int_\Gamma g ds = 0 \tag{2.5}$$

Combining Theorem 2.1 of [4] and Theorem 2.2 above we have the following theorems.

Theorem 2.3. If $g \in B_{\hat\beta}^{1,2}(\Gamma) \cap C^0(\Gamma)$ *with* $\hat\beta = (\hat\beta_1, \hat\beta_2, \ldots, \hat\beta_M)$, $\hat\beta_i = (\hat\beta_{i,1}, \hat\beta_{i,2})$, $0 < \hat\beta_{i,j} < 1$, $1 \le i \le M$, $1 \le j \le 2$, *then the problem* (2.3) *has a unique solution* $u \in B_\beta^2(\Omega)$ *with β given by*

$$\begin{aligned} &\beta_i = \beta_i^* \text{ if } \beta_i^* > 1 - \tfrac{\pi}{\omega_i} \\ &\beta_i > 1 - \tfrac{\pi}{\omega_i} \text{ if } \beta_i^* \le 1 - \tfrac{\pi}{\omega_i} \end{aligned} \tag{2.6}$$

where β_i^* *satisfies* (2.2).

Theorem 2.4. If $g \in B_{\hat\beta}^{0,1}(\Gamma)$ *with* $\hat\beta = (\hat\beta_1, \hat\beta_2, \ldots, \hat\beta_M)$, $\hat\beta_i = (\hat\beta_{i,1}, \hat\beta_{i,2})$, $0 < \hat\beta_{i,j} < 1$, $1 \le i \le M$, $1 \le j \le 2$, *then the problem* (2.4), (2.5) *has a solution* $u \in B_\beta^2(\Omega)$ *which is unique up to a constant with β given by* (2.6).

3. Boundary Element Method for the Neumann Problem

We consider the Neumann problem (2.4) with solvability condition (2.5). As a consequence of Theorem 2.1, we obtain

Theorem 3.1. Let $u \in B_\beta^2(\Omega)$, *then* $\left.\frac{\partial u}{\partial n}\right|_\Gamma \in B_{\hat\beta}^{0,1}(\Gamma)$ *with* $\hat\beta_{i,j}$ *given by* (2.1).

A combination of Theorem 2.1 in [4] and Theorem 4.5 in [6] leads to the result:

Theorem 3.2. Let $g \in B_{\hat\beta}^{0,1}(\Gamma)$ *satisfying* (2.5) *then there exists* $u \in B_\beta^2(\Omega)$ *solving* (2.4) *with β given by* (2.6). *Furthermore* $u|_\Gamma \in B_{\tilde\beta}^{1,2}(\Gamma)$ *with $\tilde\beta$ satisfying* (2.1).

Proof. Due to Theorem 2.2 there exists a $G \in B_{\beta^*}^1(\Omega)$ such that $G|_\Gamma = g$. Then by the definition of $B_{\beta^*}^{1/2}(\Gamma)$, we have $g \in B_{\beta^*}^{1/2}(\Gamma)$ with β^* satisfying (2.2). Then by Theorem 3.2 in [4] problem (2.4) has a unique solution $u \in B_\beta^2(\Omega)$ (up to a constant) with β satisfying (2.6). Then applying Theorem 2.1 we have $u|_\Gamma \in B_{\tilde\beta}^{1,2}(\Gamma)$ with $\tilde\beta$ given by (2.1). $\square$

Remark 3.1. In general, we have $\tilde{\beta} = \hat{\beta}+\epsilon$ with some $\epsilon > 0$. According to (2.1), (2.2), and (2.6) $\tilde{\beta}_{i,1}$ (resp. $\tilde{\beta}_{i,2}$) depends on the interior angle ω_i, and on $\hat{\beta}_i$ as well as $\hat{\beta}_{i-1}$ (resp. $\hat{\beta}_{i+1}$). For instance, if $g \in B^0_{\hat{\beta}_i}(\Gamma_i) \cap B^0_{\hat{\beta}_{i-1}}(\Gamma_{i-1})$, $0 < \hat{\beta}_{i-1}, \hat{\beta}_i < \frac{1}{2}$, we have $\beta^*_i = \max(\tilde{\beta}_{i,1}, \tilde{\beta}_{i-1,2}) + \frac{1}{2}$ by (2.2). Since $0 < \omega_i < 2\pi$, $1 - \frac{\pi}{\omega_i} < \frac{1}{2} < \beta^*_i$ and $\beta_i = \beta^*_i$ by (2.6). Due to (2.1) $\tilde{\beta}_{i,1} \in (\beta_i - 1, \frac{1}{2})$, hence $\tilde{\beta}_{i,1} > \beta_i - \frac{1}{2} = \max(\tilde{\beta}_{i,1}, \tilde{\beta}_{i-1,2}) \geq \hat{\beta}_{i,1}$. For the other cases the relations between $\tilde{\beta}_i$, $\hat{\beta}_i$, $\hat{\beta}_{i-1}$ (resp. $\hat{\beta}_{i+1}$), and ω_i can be derived similarly.

Next we derive a boundary integral equation to solve the Neumann problem (2.4). Inserting the fundamental solution $v = -\frac{1}{2\pi}\,\ell n|x-y|$ of $\Delta u = 0$ into the second Green formula

$$\int_\Omega (\Delta u v - u \Delta v)dx = \int_\Gamma (u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n})ds \tag{3.1}$$

we get the representation formula for $x \in \Omega$

$$u(x) = \frac{1}{2\pi}\int_\Gamma \frac{\partial}{\partial n_y}\,\ell n|x-y|u(y)ds(y) - \frac{1}{2\pi}\int_\Gamma \frac{\partial u(y)}{\partial n_y}\,\ell n|x-y|ds(y) \tag{3.2}$$

which yields on Γ the integral equation

$$Du = (1-K')\frac{\partial u}{\partial n} \tag{3.3}$$

with the integral operators for $x \in \Gamma$

$$Du(x) = -\frac{1}{\pi}\frac{\partial}{\partial n_x}\int_\Gamma \frac{\partial}{\partial n_y}\,\ell n|x-y|u(y)ds(y),$$

and

$$K'\frac{\partial u}{\partial n}(x) = -\frac{1}{\pi}\frac{\partial}{\partial n_x}\int_\Gamma \ell n|x-y|\,\frac{\partial u(y)}{\partial n_y}\,ds(y).$$

Now, equation **(3.3)** and the boundary conditions in (2.4) lead to the first kind integral equation (see [12])

$$Du = f \text{ on } \Gamma \tag{3.4}$$

where $f = (1-K')g$. There holds the following result:

Theorem 3.3. *For given* $g \in B^{0,1}_{\hat{\beta}}(\Gamma)$ *satisfying* (2.5) *with* $\hat{\beta}$ *as in Theorem* 2.4, *the integral equation*

(3.4) *together with the side condition*

$$\int_{\Gamma} u\, ds = 0 \qquad (3.5)$$

has a unique solution $u \in B^{1,2}_{\tilde{\beta}}(\Gamma)$ *with* $\tilde{\beta}$ *determined by* (2.1) *and* (2.6).

Proof. First from Remark 3.2 and Lemma 3.2, below, we have $g \in H^{-1/2}(\Gamma)$.

By Theorem 1.5 (ii) of [12] together with uniqueness of the system (3.4), (3.5), its solution u exists uniquely in $H^{1/2}(\Gamma)$. Inserting this u and the boundary condition $g = \frac{\partial u}{\partial n}$ into (3.2) we define for $x \in \Omega$

$$\tilde{u}(x) = \frac{1}{2\pi}\int_{\Gamma} \frac{\partial}{\partial n_y} \ell n|x-y|u(y)ds(y) - \frac{1}{2\pi}\int_{\Gamma} g(y)\ell n|x-y|ds(y). \qquad (3.6)$$

By [12] $\tilde{u} \in H^1(\Omega)$ solves the Neumann problem (2.4). Due to the derivation above $v = \tilde{u}|_{\Gamma} \in H^{1/2}(\Gamma)$ solves the integral equation (3.4).

On the other hand from Theorem 3.2 we know that for $g \in B^{0,1}_{\tilde{\beta}}(\Gamma)$ there exists a unique solution (up to a constant) $U \in B^2_{\beta}(\Omega)$ of the Neumann problem (2.4) with β satisfying (2.6). But from the definition of $B^2_{\beta}(\Omega)$ we have $U \in H^1(\Omega)$. Hence $\tilde{u} - U$ = constant on Ω. But for $v = \tilde{u}|_{\Gamma}$ we have $u - v$ = constant on Γ where $u \in H^{1/2}(\Gamma)$ solves the integral equation (3.4). Hence $u = U|_{\Gamma} + c$, c = constant, and since $U|_{\Gamma} \in B^{1,2}_{\tilde{\beta}}(\Gamma)$ due to Theorem 3.2 for $\tilde{\beta}$ satisfying (2.1), the assertion $u \in B^{1,2}_{\tilde{\beta}}(\Gamma)$ follows. □

In the proof of Theorem 3.3 we made use of the following results.

Lemma 3.1. *Let* $g \in H^{0,0}_{\dot{\tilde{\beta}}}(I)$ *with* $0 < \dot{\tilde{\beta}} < \frac{1}{2}$, *then* $g \in (H^{1/2}(I))'$.

Proof. We may assume that $I = [0,1]$, and $\Phi_{\dot{\tilde{\beta}}} = x^{\dot{\tilde{\beta}}_1}$, $x \in I$, $0 < \dot{\tilde{\beta}}_1 < \frac{1}{2}$. For any $v \in H^{1/2}(I)$

$$\left|\int_I gv dx\right| \le \left(\int_I |x^{\dot{\tilde{\beta}}_1} g|^2 dx\right)^{1/2}\left(\int_I |x^{-2\dot{\tilde{\beta}}_1} v|^2 dx\right)^{1/2} \le ||g||_{H^{0,0}_{\dot{\tilde{\beta}}}(I)}\left(\int_I x^{-2\dot{\tilde{\beta}}_1}|v|^2 dx\right)^{1/2}$$

Since $0 < \dot{\tilde{\beta}}_1 < \frac{1}{2}$, there exist integers p,q such that $\frac{1}{p} + \frac{1}{q} = 1$, and $2\dot{\tilde{\beta}}_1 p < 1$. Therefore by the imbedding theorem (see, e.g., [16])

$$\int_I x^{-2\dot{\tilde{\beta}}_1}|v|^2 dx \le \left(\int_I x^{-2\dot{\tilde{\beta}}_1 p} dx\right)^{1/p}\left(\int |v|^{2q} dx\right)^{1/q} \le C||v||^2_{L_{2q}(I)} \le C||v||_{H^{1/2}(I)}$$

and

$$|\int_I gv\,dx| \le C\|g\|_{H^{0,0}_{\hat\beta}(I)}\|v\|_{H^{1/2}(I)}$$

which leads to the conclusion. □

Lemma 3.2. *If* $g \in H^{0,0}_{\hat\beta}(\Gamma)$, $0 < \hat\beta < \frac{1}{2}$, *then* $g \in H^{-1/2}(\Gamma)$.

Proof. First from [26] we observe that for $v \in H^{1/2}(\Gamma)$ there holds

$$\sum_{i=1}^{M}\|v\|_{H^{1/2}(\Gamma_i)} \le C\|v\|_{H^{1/2}(\Gamma)}$$

with a constant C independent of v and the length of Γ_i. Hence we obtain for $g \in H^{0,0}_{\hat\beta}(\Gamma)$

$$\|g\|_{H^{-1/2}(\Gamma)} := \frac{\int_\Gamma gv\,ds}{\|v\|_{H^{1/2}(\Gamma)}} \le C\sum_{i=1}^{M}\frac{\int_{\Gamma_i} gv\,ds}{\|v\|_{H^{1/2}(\Gamma_i)}} \le C\sum_{i=1}^{M}\|g\|_{H^{0,0}_{\hat\beta}(\Gamma_i)} \le C\|g\|_{H^{0,0}_{\hat\beta}(\Gamma)}$$

with a constant C. Here we have made use of Lemma 3.1 and taken $\Gamma_i = I$. Hence by the definition of $H^{-1/2}(\Gamma)$ the assertion follows. □

Remark 3.2. Definition shows that $g \in B^{0,1}_{\hat\beta}(\Gamma)$ belongs to $B^{0}_{\hat{\hat\beta}}(\Gamma)$ with

$$\hat{\hat\beta}_i = \begin{cases} \hat\beta_i & \text{for } 0 < \hat\beta_i < \frac{1}{2} \\ \hat\beta_i - \frac{1}{2} & \text{for } \frac{1}{2} < \hat\beta_i < 1 \end{cases}$$

Furthermore we have $B^{0}_{\hat{\hat\beta}}(\Gamma) \subset H^{0,0}_{\hat{\hat\beta}}(\Gamma)$ due to the definition of $B^{0}_{\hat{\hat\beta}}(\Gamma)$ and therefore by Lemma 3.2 we have $g \in B^{0,1}_{\hat\beta}(\Gamma)$ belongs to $H^{-1/2}(\Gamma)$.

Now we discuss the numerical solution of the system (3.4), (3.5) by the h-p version of the Galerkin boundary element method.

Let Ω be an L-shaped domain as shown in Fig. 3.1. We assume *for simplicity* that the solution u of (3.4), (3.5) belongs to $B^{1}_{\hat\beta}(\Gamma)$ (resp. $B^{2}_{\hat\beta}(\Gamma)$) with $\hat\Phi_{\hat\beta_1} = |y|^{\hat\beta_{1,2}}$, $\hat\Phi_{\hat\beta_2} = |x|^{\hat\beta_{2,1}}$, $\frac{1}{6} < \hat\beta_{1,2}, \hat\beta_{2,1} < \frac{1}{2}$, (resp. $\frac{5}{6} < \hat\beta_{1,2}, \hat\beta_{2,1} < 1$) and $\hat\Phi_{\hat\beta_j} = 1$ for $3 \le j \le 6$,

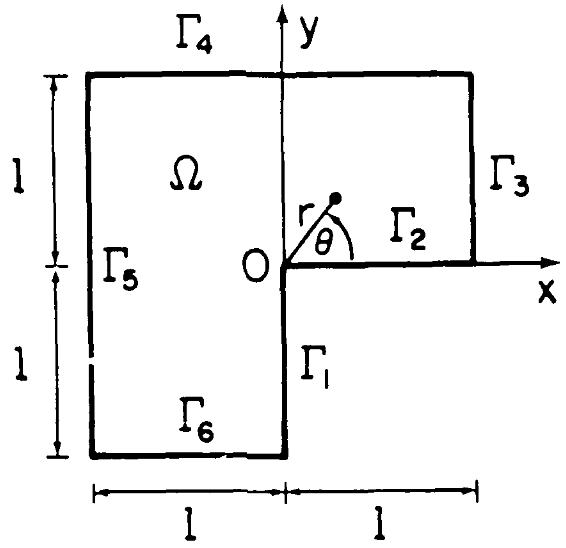
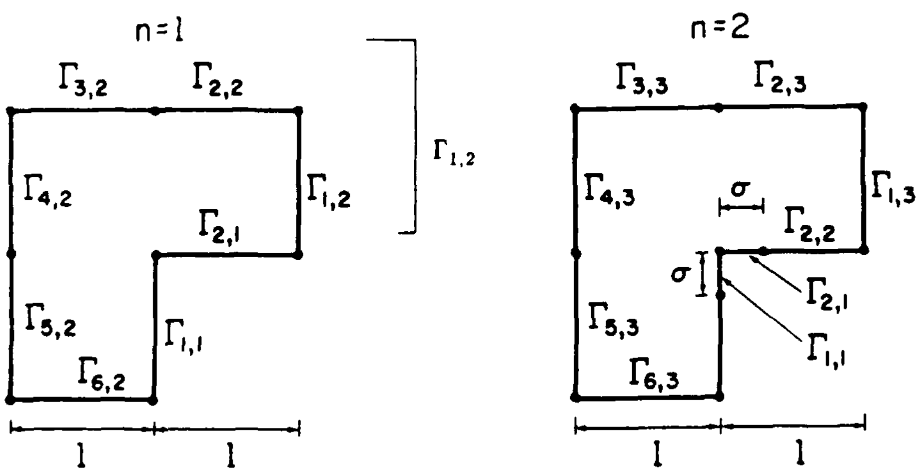


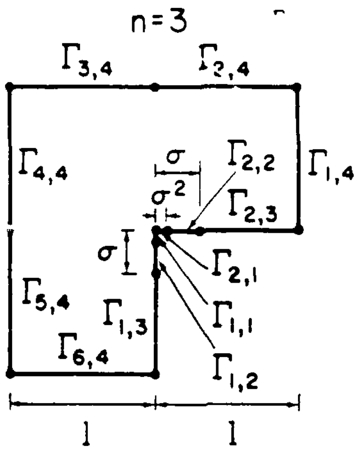
Fig. 3.1. L-Shaped Domain Ω.

i.e., the singularity occurs only at the origin. For example this is the case of $u = r^{1/3} \sin \frac{\theta}{3}$ on Γ (resp. $u = r^{2/3} \cos \frac{2}{3}\theta$) *where* (r,θ) *denote the polar coordinates centered at the origin.*

Let $\sigma \in (0,1)$ be the mesh factor and n, integer, be the number of layers, and let $\Gamma_{i,j}$, $1 \leq i \leq I(j)$, $1 \leq j \leq n+1$ be the boundary elements such that $\mathrm{dist}(0,\Gamma_{i,j}) = \sigma^{n+1-j}$, $1 < j \leq n+1$ and $\mathrm{dist}(0,\Gamma_{i,1}) = 0$, $1 \leq i \leq I(j)$. Then $\Gamma_\sigma^n = \{\Gamma_{i,j},\ 1 \leq i \leq I(j),\ 1 \leq j \leq n+1\}$ is called the geometric mesh on Γ associated with σ and n. Fig. 3.2 shows a sequence of the geometric meshes with $\sigma = 0.15$.



Mesh Γ_σ^1 Mesh Γ_σ^2



Mesh Γ_σ^3

Fig. 3.2 Geometric Mesh Γ_σ^n, n = 1,2,3, σ = 0.15.

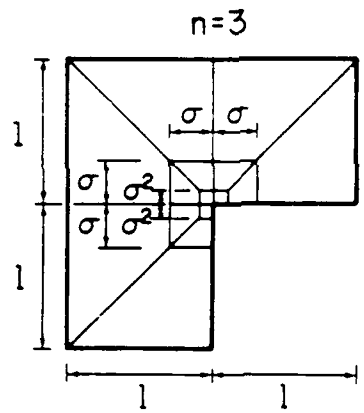


Fig. 3.3. Geometric Mesh Ω_σ^n, n = 3, σ = 0.15

Let $P = \{p_{i,j}, 1 \leq i \leq I(j), 1 \leq j \leq n+1\}$ be the degree vector with $p_{i,j} \geq 1$ integer. The boundary element space associated with the geometric mesh Γ_σ^n and degree vector P is defined as

$$S^P(\Gamma_\sigma^n) = \{\phi \mid \phi|_{\Gamma_{i,j}} \text{ is a polynomial of degree} \leq p_{i,j}\}$$

and

$$\dot{S}^P(\Gamma_\sigma^n) = S^P(\Gamma_\sigma^n) \cap C^0(\Gamma) \subset H^{1/2}(\Gamma).$$

The geometric mesh Γ_σ^n can be extended to a geometric mesh Ω_σ^n on Ω as shown in Fig. 3.3. Therefore the geoemtric mesh Γ_σ^n on $\partial\Omega$ can be defined as the trace of the geometric mesh

$$\Omega_\sigma^n = \{\Omega_{i,j}, 1 \le i \le I(j), 1 \le j \le n+1\},$$

i.e., $\Gamma_{i,j} = \partial\Omega_{i,j} \cap \partial\Omega$. Hence the space $S^p(\Gamma_\sigma^n)$ is actually the trace spaces of $S^p(\Omega_\sigma^n)$, i.e.,

$$S^p(\Gamma_\sigma^n) = \{\phi|_\Gamma : \phi \in S^p(\Omega_\sigma^n)\}$$

and

$$\mathring{S}^p(\Gamma_\sigma^n) = \{\phi|_\Gamma : \phi \in S^p(\Omega_\sigma^n) \cap C^0(\bar{\Omega})\}.$$

For details of the definition of the geometric mesh on a curvilinear polygonal domain Ω and the finite element space $S^p(\Omega_\sigma^n)$, see [5], [7], [17].

From [17] we quote the following approximation property of $S^p(\Omega_\sigma^n)$.

Lemma 3.1. *Let* $u \in B_\beta^2(\Omega)$ *with* $\Phi_\beta = r^\beta$, $0 < \beta < 1$. *For any* $\sigma \in (0,1)$ *there exists* $w \in S^p(\Omega_\sigma^n) \cap C^0(\bar{\Omega})$ *with* $p_{i,j} = p_j \ge 1$ *and* $j\mu \le p_j \le n\nu$, $0 \le \mu \le \nu < \infty$, *which satisfies*

$$||u-w||_{H^1(\Omega)} \le Ce^{-bn}$$

where n *is the number of layers of the geometric mesh* Ω_σ^n, *and* C,b *are some constants depending on* β *and* σ *but not on* n.

By the trace theorem we know that $\mathring{S}^p(\Gamma_\sigma^n) \subset H^{1/2}(\Gamma)$. The corresponding *boundary element Galerkin procedure* for the integral equation (3.4), (3.5) reads: *For given* $g \in B_\beta^{0,1}(\Gamma)$ *find* $u_p \in \mathring{S}^p(\Gamma_\sigma^n)$ *such that*

$$\left\langle Du_p, w\right\rangle_{L^2(\Gamma)} = \left\langle (1-K')g, w\right\rangle_{L^2(\Gamma)}, \ \forall\, w \in \mathring{S}^p(\Gamma_\sigma^n) \tag{3.7}$$

and

$$\int_\Gamma u_p ds = 0$$

where $\langle\cdot,\cdot\rangle$ denotes the duality between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$. We have the following approximation result of the boundary element Galerkin method.

Theorem 3.4. *Let* $u \in B_\beta^1(\Gamma)$ *(resp.* $B_\beta^2(\Gamma)$*) be the solution of the integral equation* (3.4), *and* Γ *be the boundary of the L-shaped domain as shown in Fig.* 3.1, *where* $\hat{\beta}_{i,j} = \hat{\beta}_{1,1} = \hat{\beta}_{2,2} = 0$, $3 \le i \le 6$, $j = 1,2$ *and* $\frac{1}{6} \le \hat{\beta}_{1,2}, \hat{\beta}_{2,1} < \frac{1}{2}$ *(resp.* $\frac{5}{6} < \hat{\beta}_{1,2}, \hat{\beta}_{2,1} < 1$*). Let* Γ_σ^n, $\sigma \in (0,1)$ *be the geometric mesh* *on* Γ: *and let* $\dot{S}^p(\Gamma_\sigma^n)$ *denote the boundary element space defined above with* $p_{i,j} = p_j \geq 1$, $i\nu \leq p_j \leq n\nu$, $0 \leq \mu \leq \nu < \infty$. *Then the boundary element Galerkin solution* u_p *of* (3.7) *converges to* u *in* $H^{1/2}(\Gamma)$ *exponentially, i.e.,*

$$||u-u_p||_{H^{1/2}(\Gamma)} \leq Ce^{-bN^{1/2}} \tag{3.8}$$

where N *is the number of degrees of freedom,* C *and* b *are some constants depending on* σ *and* β *but not on* N.

Proof. Note the operator D in (3.4) is strongly elliptic in $H^{1/2}(\Gamma)$, i.e., satisfies a Gårding inequality in $H^{1/2}(\Gamma)$ (cf. Theorem 1.15 (ii) in [12]). Hence due to [19], [27] any conforming Galerkin scheme converges in $H^{1/2}(\Gamma)$, and we have

$$||u-u_p||_{H^{1/2}(\Gamma)} \leq C \inf_{w_p \in \dot{S}^p(\Gamma_\sigma^n)} ||u-w_p||_{H^{1/2}(\Gamma)}.$$

Let $U \in B_\beta^2(\Omega)$ be the solution of the boundary value problem (2.4), (2.5), Ω_σ^n be the geometric mesh on Ω, and $S^p(\Omega_\sigma^n) \cap C^0(\bar{\Omega})$ be the 2D finite element space. By Lemma 3.1 there exists a $W_p \in S^p(\Omega_\sigma^n) \cap C^0(\bar{\Omega})$ such that

$$||U-W_p||_{H^1(\Omega)} \leq C_1 e^{-b_1 n} \tag{3.9}$$

where n is the number of layers of Ω_σ^n.

Let $w_p = W_p|_\Gamma$, then $w_p \in \dot{S}^p(\Gamma_\sigma^n)$, and by the trace theorem

$$||u-w_p||_{H^{1/2}(\Gamma)} = ||U|_\Gamma - W_p|_\Gamma||_{H^{1/2}(\Gamma)} \leq \tilde{C}||U-W_p||_{H^1(\Omega)}. \tag{3.10}$$

Now, (3.9) and (3.10) together yield

$$||u-w_p||_{H^{1/2}(\Gamma)} \leq Ce^{-b_1 n}.$$

Note that I(j) $\leq$ k uniformly with respect to j, $1 \leq j \leq n+1$. Hence $N \leq kn^2$, where k is some constant independent of N, which leads to (3.8) immediately. □

4. Boundary Element Method for the Dirichlet Problem

In this section we consider the Dirichlet problem (2.3).

Lemma 4.1. *If* $g \in B_{\beta}^{1,2}(\Gamma)$, *then* $g \in H^{1/2}(\Gamma)$.

Proof. Since $g \in B_{\beta}^{1,2}(\Gamma)$, $g|_{\Gamma_i} \in H_{\beta_i}^{2,1}(\Gamma_i)$ for $0 < \beta_i < \frac{1}{2}$ or $g|_{\Gamma_i} \in H_{\beta_i}^{2,2}(\Gamma_i)$ for $\frac{1}{2} < \beta_i < 1$, $1 \leq i \leq M$. By Theorem 2.2 there exists a $G \in H_{\beta^*}^{2,2}(\Omega)$ with β^* satisfying (2.2) such that $G|_{\Gamma} = g$. By the definition of $H_{\beta^*}^{2,2}(\Omega)$, $G \in H^1(\Omega)$. Therefore $g \in H^{1/2}(\Gamma)$. □

Theorem 4.1. *Let* $g \in B_{\beta}^{1,2}(\Gamma) \cap C^0(\Gamma)$. *Then there exists a unique solution* $u \in B_{\beta}^{2}(\Omega)$ *of* (2.3) *with* β *satisfying* (2.6). *Further* $\left.\frac{\partial u}{\partial n}\right|_{\Gamma} \in B_{\tilde\beta}^{0,1}(\Gamma)$ *with* $\tilde\beta$ *satisfying* (2.1).

Proof. By Theorem 2.3 the problem (2.3) has a unique solution $u \in B_{\beta}^{2}(\Omega)$ with β satisfying (2.6). Therefore, $D^{\alpha}u \in B_{\beta}^{1}(\Omega)$ for $|\alpha| = 1$. Applying Theorem 2.1 we obtain $\left.\frac{\partial u}{\partial n}\right|_{\Gamma} \in B_{\tilde\beta}^{0,1}(\Gamma)$ with $\tilde\beta$ satisfying (2.1). □

Next we derive a boundary integral equation for (2.3). We introduce the integral operator

$$V\frac{\partial u}{\partial n}(x) = -\frac{1}{\pi}\int_{\Gamma}\frac{\partial u(y)}{\partial n_y}\,\ell n|x-y|\,ds(y), \quad x \in \Gamma \tag{4.1}$$

and

$$Ku(x) = -\frac{1}{\pi}\int_{\Gamma}\frac{\partial}{\partial n_y}\,\ell n|x-y|\,u(y)\,ds(y), \quad x \in \Gamma \tag{4.2}$$

Taking the limit $x \in \Gamma$ in (3.2) we obtain via the well-known jump relation for the double layer potential K the integral equation on Γ

$$V\frac{\partial u}{\partial n}(x) = (1+K)u(x), \quad x \in \Gamma. \tag{4.3}$$

Insertion of the boundary condition of (2.3) into (4.3) leads to the first kind integral equation

$$V\frac{\partial u}{\partial n} = f \text{ on } \Gamma \tag{4.4}$$

with $f = (1+K)g$ for which there holds the following result.

Theorem 4.2. Let $\mathrm{cap}(\Gamma) \neq 1$ *where* $\mathrm{cap}(\Gamma)$ *is the capacity (or conformal radius) of* Γ. *Then for given* $g \in \overset{1,2}{B_{\tilde\beta}}(\Gamma) \cap C^0(\Gamma)$ *there exists exactly one solution* $\frac{\partial u}{\partial n} \in \overset{0,1}{B_{\tilde\beta}}(\Gamma)$ *of the integral equation* (4.4) *with* $\tilde\beta$ *given by* (2.1) *and* (2.6).

Proof. From Lemma 4.1 we have that $g \in \overset{1,2}{B_{\tilde\beta}}(\Gamma)$ implies $g \in H^{1/2}(\Gamma)$. By Theorems 3.5, 3.8, and 3.9 in [13] the integral equation (4.4) has a unique solution $\frac{\partial u}{\partial n} \in H^{-1/2}(\Gamma)$ for given data $g \in H^{1/2}(\Gamma)$. Then inserting this $\frac{\partial u}{\partial n}$ together with the given data $g = u$ into (3.2) we define

$$\tilde u(x) = \frac{1}{2\pi}\int_\Gamma \frac{\partial}{\partial n_y}\ell n|x-y|g(y)ds(y) - \frac{1}{2\pi}\int_\Gamma \frac{\partial u}{\partial n}(y)\ell n|x-y|ds(y), \quad \forall\, x \in \Omega. \tag{4.5}$$

By Theorem 3.9 in [13] we know $\tilde u \in H^1(\Omega)$ and solves the Dirichlet boundary value problem (2.3). Also $\left.\frac{\partial \tilde u}{\partial n}\right|_\Gamma \in H^{-1/2}(\Gamma)$ satisfies the integral equation (4.4) due to the definition of $\tilde u$. Due to the uniqueness of the solution of (4.4) in $H^{-1/2}(\Gamma)$ we have $\frac{\partial u}{\partial n} = \frac{\partial \tilde u}{\partial n}$.

By Theorem 4.1 there exists a unique solution $U(x) \in B^2_\beta(\Omega)$ of (2.3) with β satisfying (2.6), and $\left.\frac{\partial U}{\partial n}\right|_\Gamma \in \overset{0,1}{B_{\tilde\beta}}(\Gamma)$ for $\tilde\beta$ satisfying (2.1). Since $B^2_\beta(\Omega) \subset H^1(\Omega)$ we have $\tilde u = U$ by the uniqueness of the solution of the Dirichlet problem. Hence $\left.\frac{\partial \tilde u}{\partial n}\right|_\Gamma = \left.\frac{\partial U}{\partial n}\right|_\Gamma \in \overset{0,1}{B_{\tilde\beta}}(\Gamma)$. But we also have $\left.\frac{\partial \tilde u}{\partial n}\right|_\Gamma = \frac{\partial u}{\partial n}$ completing the theorem. $\square$

Now we consider the rate of convergence for the h-p version of the boundary element Galerkin method for the integral equation (4.4). For simplicity we assume again that Ω is the L-shaped domain shown in Fig. 3.1 with $\mathrm{cap}(\Gamma) \neq 1$ and that g and the solution $\frac{\partial u}{\partial n}$ of (4.4) have a singularity at the origin only. Then the geometric mesh Γ^n_σ on Γ and the boundary element space $S^p(\Gamma^n_\sigma)$ are defined as in the previous section. They are the traces of the geometric mesh Ω^n_σ on Ω and of the finite element space $S^p(\Omega^n_\sigma)$, respectively. Obviously $S^{p-1}(\Gamma^n_\sigma) \subset L_2(\Gamma) \subset H^{-1/2}(\Gamma)$.

The ***Galerkin procedure*** for the integral equation (4.4) reads: *For given* $g \in \overset{1,2}{B_{\tilde\beta}}(\Gamma) \cap C^0(\Gamma)$ *find* $\psi_p \in S^{p-1}(\Gamma^n_\sigma)$ *such that for all* $\phi_p \in S^{p-1}(\Gamma^n_\sigma)$

$$\left\langle V\psi_p, \phi_p\right\rangle_{L^2(\Gamma)} = \left\langle (1+K)g, \phi_p\right\rangle_{L^2(\Gamma)}. \tag{4.6}$$

For the boundary element solution ψ_p we have the following approximation theorem.

Theorem 4.3. Let $\frac{\partial u}{\partial n} \in B^0_\beta(\Gamma)$ *(resp.* $B^1_\beta(\Gamma)$*) be the solution of the integral equation* (4.6) *where* Γ *with* cap(Γ) $\neq 1$ *is the boundary of the L-shaped domain as shown in Fig.* 3.1. *and* $\beta_{1,1} = \beta_{1,2} = \beta_{i,j}$, $3 \leq i \leq 6$, $j = 1,2$, $\frac{1}{6} < \beta_{1,2}, \beta_{2,1} < \frac{1}{2}$ (*resp.* $\frac{5}{6} < \beta_{1,2}, \beta_{2,1} < 1$). *Let* Γ^n_σ, $\sigma \in (0,1)$ *be the geometric mesh on* Γ *and* $S^{p-1}(\Gamma^n_\sigma)$ *be the boundary element space defined in previous sections with* $p_{i,j} = p_j \geq 1$, $j\mu \leq p_j \leq \nu n$, $0 \leq \mu \leq \nu < \infty$. *Then the boundary element Galerkin solution* ψ_p *of* (4.6) *converges to* $\frac{\partial u}{\partial n}$ *in* $H^{-1/2}(\Gamma)$ *exponentially, i.e.,*

$$\left\|\psi_p - \frac{\partial u}{\partial n}\right\|_{H^{-1/2}(\Gamma)} \leq Ce^{-bN^{1/2}} \tag{4.7}$$

where N *is the number of degrees of freedom, and* C, b *are some constants depending on* σ *and* β, *but not on* N.

Proof. Since the operator V is strongly elliptic, i.e., satisfies a Gårding inequality in $H^{-1/2}(\Gamma)$ (cf., Theorem 2.19 in [13]), any conforming Galerkin scheme converges, and we have

$$\left\|\frac{\partial u}{\partial n} - \psi_p\right\|_{H^{-1/2}(\Gamma)} \leq C \inf_{w_p \in S^{p-1}(\Gamma^n_\sigma)} \left\|\frac{\partial u}{\partial n} - w_p\right\|_{H^{-1/2}(\Gamma)}. \tag{4.8}$$

Let V be the harmonic conjugate of the solution U of the boundary value problem (2.3). Then V satisfies

$$\begin{aligned} &\Delta V = 0 \text{ in } \Omega \\ &\left.\frac{\partial V}{\partial s}\right|_\Gamma = \left.\frac{\partial U}{\partial n}\right|_\Gamma =: \mathbf{k} \end{aligned} \tag{4.9}$$

and $V \in B^2_\beta(\Omega)$. Following the proof of Theorem 5.1 of [7] we can show that the 2D *finite element solution* $V_p \in S^p(\Omega^n_\sigma) \cap C^0(\bar{\Omega})$ of the Galerkin equations for (4.9) satisfies

$$||V - V_p||_{H^1(\Omega)} \leq C_1 e^{-b_1 n} \tag{4.10}$$

where C_1, b_1 are independent of n and p. Furthermore

$$v_p := \left.\frac{\partial V_p}{\partial s}\right|_\Gamma = \mathbf{k}_p \in S^{p-1}(\Gamma^n_\sigma),$$

where $\mathbf{k}_p$ is the projection of k on $S^{p-1}(\Gamma^n_\sigma)$. Therefore we have with $\frac{\partial u}{\partial n} = \left.\frac{\partial U}{\partial n}\right|_\Gamma$ (compare Theorem

4.2) and using (4.9)

$$\left\|\frac{\partial u}{\partial n} - v_p\right\|_{H^{-1/2}(\Gamma)} = \left\|\frac{\partial U}{\partial n}\Big|_\Gamma - \frac{\partial V_p}{\partial s}\Big|_\Gamma\right\|_{H^{-1/2}(\Gamma)} = \left\|\left(\frac{\partial V}{\partial s} - \frac{\partial V_p}{\partial s}\right)\Big|_\Gamma\right\|_{H^{-1/2}(\Gamma)} \tag{4.11}$$

From Lemma 3.2 in [25] it follows that $w \in H^{1/2}(\Gamma)$ implies for the derivative $w' \in H^{-1/2}(\Gamma)$. Hence there exists a constant C such that

$$\left\|\frac{\partial V}{\partial s} - \frac{\partial V_p}{\partial s}\right\|_{H^{-1/2}(\Gamma)} \leq C\|V - V_p\|_{H^{1/2}(\Gamma)}.$$

Therefore together with the trace theorem we obtain from (4.10) and (4.11)

$$\left\|\frac{\partial u}{\partial n} - v_p\right\|_{H^{-1/2}(\Gamma)} \leq C\|V - V_p\|_{H^1(\Omega)} \leq C_1 e^{-b_1 n}. \tag{4.12}$$

Now, (4.12) together with (4.8) yields (4.7) by noting that $N \leq kn^2$ for some constant k independent of N. □

Remark 1. The regularity of the solutions of the boundary integral equations for mixed boundary value problems and the exponential rate of convergence for the h-p version of the boundary element Galerkin method can be proven similarly, but we will not elaborate it here, see [4], [6], [5], [7], [13].

Remark 2. In previous sections we assumed that Ω is a straight-line polygon. If Ω is a curvilinear polygon with a piecewise analytic boundary, the solutions of the boundary value problems (2.3) and (2.4) belong to $B^2_{\beta+\epsilon}(\Omega)$ for any $\epsilon > 0$, (see [5]). Then it could be proven that the solutions of equations (3.4) and (4.4) are in $B^{1,2}_{\tilde\beta+\epsilon}(\Gamma)$ and $B^{0,1}_{\tilde\beta+\epsilon}(\Gamma)$ respectively. Thus Theorems 3.4 and 4.3 remain valid.

Remark 3. The geometric mesh shown in Fig. 3.2 is designed for the problems with a singularity at only one corner. But it is not difficult to generalize the above technique to the problem with singularities at each corner of Γ, and the exponential rate of convergence of boundary element Galerkin solution can be proven again.

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